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Generalized Van der Waerden identities

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Abstract. The idea of generalized Van der Waerden identities is introduced to facilitate the polynomial expansion of multi-spin functions of the type that are frequently encountered in theoretical studies of Ising spin systems.

In the literature concerning spin systems, operator identities for $\exp(\lambda S_z)$ exist that are often termed the 'Van der Waerden identities'. For spin $\frac{1}{2}$ and spin 1, for example, they are

$$\exp(\lambda S_z) = \cosh(\lambda/2) + 2S_z \sinh(\lambda/2) \tag{1}$$

and

$$\exp(\lambda S_z) = 1 - S_z^2 + S_z^2 \cosh(\lambda) + S_z \sinh(\lambda) \tag{2}$$

respectively.

These relations, together with the differential operator method, have been extensively employed in effective field theories of Ising systems to express a general function of $\sum_i S_{iz}$ of spin operators as a finite polynomial in the S_{iz}. For example, in the effective field treatment of the spin-$\frac{1}{2}$ Ising model with nearest-neighbour interactions on a lattice with coordination number, q, one encounters the function

$$g = \tanh\left(\tfrac{1}{2}\beta J \sum_{i=1}^{q} S_{iz}\right). \tag{3}$$

To express this as a finite polynomial in the S_{iz}, the differential operator technique,

$$f(\alpha) = \exp(\alpha D) f(x)|_{x\to 0} \tag{4}$$

with $D \equiv \partial/\partial x$, can be employed [1] to give

$$g = \prod_{i=1}^{q} [\exp(\tfrac{1}{2}\beta J S_{iz} D)] g(x)|_{x\to 0} \tag{5}$$

with $g(x) = \tanh(x)$. The Van der Waerden identity (1) is then used to obtain,

$$g = \prod_{i=1}^{q} [\cosh(\tfrac{1}{4}\beta J D) + 2S_{iz} \sinh(\tfrac{1}{4}\beta J D)] g(x)|_{x\to 0}. \tag{6}$$

The final step is then to utilize (4) in the reverse direction. Whilst this procedure can in practice be carried out for general spin, it becomes increasingly very tedious to implement for

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larger spin values as the Van der Waerden identities for $\exp(\lambda S_z)$ become progressively more complicated, the greater the spin. For $S > \frac{1}{2}$ there can also be the additional complication that the function of interest contains not only $\sum_i S_{iz}$ but also $\sum_i S_{iz}^p$, with p an integer, $\leqslant 2S$. For example, in the BEG spin-1 model with biquadratic interactions the function of interest in effective field theory also depends on $\sum_i S_{iz}^2$. In that case [2,3], an additional Van der Waerden type identity,

$$\exp(\lambda S_z^2) = 1 + S_z^2[\exp(\lambda) - 1] \tag{7}$$

is also required.

In the site-diluted Ising model treated within effective field theory [4], similar expressions to (3) are encountered but with the additional complication that each S_{iz} is accompanied by the site occupancy number, c_i, that adopts values unity or zero depending on whether the site is occupied or not. In developing the theory the same procedure as above is customarily followed together with use of the additional identity $\exp(ac_i) = 1 - c_i + c_i \exp(a)$.

The purpose of our note is to introduce the concept of *generalized Van der Waerden identities* that apply to a general function $F(S_{iz})$, rather than to just the function $\exp(\lambda S_z)$. The availability of these generalized identities removes the necessity of using the differential operator technique for the purpose that has just been described, and will thus facilitate future calculations, particularly on systems with large spin values.

Consider a general function $F(S_{iz})$ involving the z-component of a single spin, S_i, of magnitude $\sqrt{[S(S+1)]}$. Because the spin has a finite set of bases states, one can expand the function as follows:

$$F(S_{iz}) = (a_0 + a_1 S_{iz} + a_2 S_{iz}^2 + \ldots + a_{2S} S_{iz}^{2S}). \tag{8}$$

Taking the expectation value of this equation with respect to the $(2S+1)$ eigenstates of S_{iz} in turn, the following equations are generated:

$$a_0 + Sa_1 + S^2 a_2 + \ldots + S^{2S} a_{2S} = F(S)$$

$$a_0 + (S-1)a_1 + (S-1)^2 a_2 + \ldots + (S-1)^{2S} a_{2S} = F(S-1)$$

$$\ldots$$

$$\ldots$$

$$a_0 - Sa_1 + S^2 a_2 + \ldots + (-2S)^{2S} a_{2S} = F(-S). \tag{9}$$

On solving these equations for the quantities a_j, one finds, from (8), the following results:

Spin $\frac{1}{2}$:

$$F(S_{iz}) = \tfrac{1}{2}[(1 + 2S_{iz})F(\tfrac{1}{2}) + (1 - 2S_{iz})F(-\tfrac{1}{2})]. \tag{10}$$

Spin 1:

$$F(S_{iz}) = \tfrac{1}{2}(S_{iz} + S_{iz}^2)F(1) + (1 - S_{iz}^2)F(0) + \tfrac{1}{2}(-S_{iz} + S_{iz}^2)F(-1). \tag{11}$$

Spin $\frac{3}{2}$:

$$F(S_{iz}) = \tfrac{1}{48}\{(-3 - 2S_{iz} + 12S_{iz}^2 + 8S_{iz}^3)F(\tfrac{3}{2}) + (27 + 54S_{iz} - 12S_{iz}^2 - 24S_{iz}^3)F(\tfrac{1}{2}) + (27 - 54S_{iz} - 12S_{iz}^2 + 24S_{iz}^3)F(-\tfrac{1}{2}) + (-3 + 2S_{iz} + 12S_{iz}^2 - 8S_{iz}^3)F(-\tfrac{3}{2})\}. \tag{12}$$

Spin 2:

$$F(S_{iz}) = \tfrac{1}{24}\{(-2S_{iz} - S_{iz}^2 + 2S_{iz}^3 + S_{iz}^4)F(2) + 4(4S_{iz} + 4S_{iz}^2 - S_{iz}^3 - S_{iz}^4)F(1) + 6(4 - 5S_{iz}^2 + S_{iz}^4)F(0) + 4(-4S_{iz} + 4S_{iz}^2 + S_{iz}^3 - S_{iz}^4)F(-1) + (2S_{iz} - S_{iz}^2 - 2S_{iz}^3 + S_{iz}^4)F(-2)\}. \tag{13}$$

Spin $\frac{5}{2}$:

$$\begin{aligned} F(S_{iz}) = \tfrac{1}{3840}\{&[45 + 18S_{iz} - 200S_{iz}^2 - 80S_{iz}^3 + 80S_{iz}^4 + 32S_{iz}^5]F(\tfrac{5}{2}) \\ &+ 5[-75 - 50S_{iz} + 312S_{iz}^2 + 208S_{iz}^3 - 48S_{iz}^4 - 32S_{iz}^5]F(\tfrac{3}{2}) \\ &+ 10[225 + 450S_{iz} - 136S_{iz}^2 - 272S_{iz}^3 + 16S_{iz}^4 + 32S_{iz}^5]F(\tfrac{1}{2}) \\ &+ 10[225 - 450S_{iz} - 136S_{iz}^2 + 272S_{iz}^3 + 16S_{iz}^4 - 32S_{iz}^5]F(-\tfrac{1}{2}) \\ &+ 5[-75 + 50S_{iz} + 312S_{iz}^2 - 208S_{iz}^3 - 48S_{iz}^4 + 32S_{iz}^5]F(-\tfrac{3}{2}) \\ &+ [45 - 18S_{iz} - 200S_{iz}^2 + 80S_{iz}^3 + 80S_{iz}^4 - 32S_{iz}^5]F(-\tfrac{5}{2})\}. \end{aligned} \tag{14}$$

These are the *generalized Van der Waerden identities* for the general function $F(S_{iz})$. In the special case, when $F(S_{iz})$ is just the function $\exp(\lambda S_{iz})$, the identities of (1) and (2) are recovered for spin $\frac{1}{2}$ and 1 respectively. For spin $\frac{3}{2}$, one finds

$$\exp(\lambda S_{iz}) = \tfrac{1}{8}[9\cosh(\lambda/2) - \cosh(3\lambda/2)] + \tfrac{1}{12}[27\sinh(\lambda/2) - \sinh(3\lambda/2)]S_{iz} + \tfrac{1}{2}[\cosh(3\lambda/2) - \cosh(\lambda/2)]S_{iz}^2 + \tfrac{1}{3}[\sinh(3\lambda/2) - 3\sinh(\lambda/2)]S_{iz}^3 \tag{15}$$

as reported in [5], and those for spin 2 and spin $\frac{5}{2}$ are, respectively,

$$\exp(\lambda S_z) = 1 + \tfrac{1}{6}[8\sinh(\lambda) - \sinh(2\lambda)]S_z + \tfrac{1}{12}[16\cosh(\lambda) - \cosh(2\lambda) - 15]S_z^2 + \tfrac{1}{6}[\sinh(2\lambda) - 2\sinh(\lambda)]S_z^3 + \tfrac{1}{12}[3 + \cosh(2\lambda) - 4\cosh(\lambda)]S_z^4 \tag{16}$$

and

$$\begin{aligned} \exp(\lambda S_z) = {} & \tfrac{1}{128}[150\cosh(\lambda/2) - 25\cosh(3\lambda/2) + 3\cosh(5\lambda/2)] \\ & \tfrac{1}{960}[2250\sinh(\lambda/2) - 125\sinh(3\lambda/2) + 9\sinh(5\lambda/2)]S_z \\ & \tfrac{1}{48}[-34\cosh(\lambda/2) + 39\cosh(3\lambda/2) - 5\cosh(5\lambda/2)]S_z^2 \\ & \tfrac{1}{24}[-34\sinh(\lambda/2) + 13\sinh(3\lambda/2) - \sinh(5\lambda/2)]S_z^3 \\ & \tfrac{1}{24}[2\cosh(\lambda/2) - 3\cosh(3\lambda/2) + \cosh(5\lambda/2)]S_z^4 \\ & \tfrac{1}{60}[10\sinh(\lambda/2) - 5\sinh(3\lambda/2) + \sinh(5\lambda/2)]S_z^5. \end{aligned}$$

For the dilute situation, the identities for a general function $F(S_{iz}, c_i)$ are the same, apart from the fact that the $F(m_i)$ on the right-hand side of equations (10)–(14) are replaced as follows:

$$F(m_i) \to c_i F(m_i, 1) + (1 - c_i) F(m_i, 0). \tag{18}$$

The generalized Van der Waerden identities may be applied to multi-spin problems if they are first recast (by inspection) into delta-function operator form:

$$F(S_{iz}, c_i) = O(S_{iz}, c_i) F(S_{iz}, c_i). \tag{19}$$

For example, for spin $\frac{1}{2}$ the appropriate operator is

$$O(S_{iz}, c_i) = \tfrac{1}{2}[(1 + 2S_{iz})\vec{\delta}_{S_{iz},\frac{1}{2}} + (1 - 2S_{iz})\vec{\delta}_{S_{iz},-\frac{1}{2}}][c_i \vec{\delta}_{c_i,1} + (1 - c_i)\vec{\delta}_{c_i,0}] \tag{20}$$

as can be seen from (10) and (18). Here $\vec{\delta}$ is a forward Kronecker delta-function that operates only on functions to the right. Likewise, the operators appropriate to other spin values may be written down directly from equations (11)–(14). The importance of these operators lies in the fact that they allow multi-spin functions of the type

$$F(x_1, x_2, \ldots, x_{2S}) \equiv F\left(\lambda_1 \sum_{i=1}^{q} c_i S_{iz}, \lambda_2 \sum_{i=1}^{q} c_i S_{iz}^2, \ldots \lambda_{2S} \sum_{i=1}^{q} c_i S_{iz}^{2S}\right) \tag{21}$$

to be expanded directly through use of the relation,

$$F(x_1, x_2, \ldots, x_{2S}) = \prod_i O(S_{iz}, c_i) F(x_1, x_2, \ldots, x_{2S}). \tag{22}$$

To conclude, we have introduced the idea of generalized Van der Waerden identities that we believe will prove useful in the study of Ising spin systems. Indeed, their advantage has already been exploited by us in a study of the thermal behaviour of the order parameters in the spin-$\frac{3}{2}$ Ising model [6].

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